Math 245C Lecture 18 Notes

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1 Properties of The Fourier Transform

1.1 Properties of the Fourier transform

If $f \in L^2(\mathbb{T}^n)$ and $k \in \mathbb{Z}^n$, then $\widehat{f}(k) = \langle f, E_k \rangle = \int_{\mathbb{R}}^n f(x) e^{-2\pi i k \cdot x}\, dx$.

Definition 1.1. The **Fourier series** are

$$\sum_{k \in \Lambda} \widehat{f}(k) E_k$$

for $\Lambda \subseteq \mathbb{Z}^n$.

Definition 1.2. Let $f \in L^1(\mathbb{R}^n)$. As $E_\xi \in L^\infty(\mathbb{R}^n)$,, $\widehat{f}(\xi) = \langle f, E_\xi \rangle = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x}\, dx$. The **Fourier transform** of f at ξ is

$$(\mathcal{F} f)(\xi) = \widehat{f}(\xi).$$

Proposition 1.1. *Let $f, g \in L^1(\mathbb{R}^n)$, let $y, \eta \in \mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear map.*

1. $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$.
2. $\widehat{f} \in C_b(\mathbb{R}^n)$.
3. $\widehat{\tau_y f} = \widehat{f} \overline{E_y}$ *and* $\tau_\eta(\widehat{f}) = \widehat{f \overline{E_\eta}}$.
4. *If* $S = T^{-1}$, *then* $\widehat{f \circ T} = |\det(S)| \widehat{f} \circ S^\top$.
5. *For* $t > 0$, *set* $f_t(x) = t^{-n} f(x/t)$. *Then* $\mathcal{F}(f_t) = (\mathcal{F}(f))_t$.

Proof. 1. Let $\xi \in \mathbb{R}^n$. Then

$$\widehat{f * g}(\xi) = \int_{\mathbb{R}^n} f * g(x) e^{-2\pi i \xi \cdot x}\, dx$$

$$= \int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot x} \int_{\mathbb{R}^n} f(x-y)g(y)\,dy\,dx$$

We can use Fubini's theorem because the product of integrable functions in separate variables is integrable.

$$= \int_{\mathbb{R}^n} g(y)e^{-2\pi i\xi\cdot y} \int_{\mathbb{R}^n} f(x-y)e^{-2\pi i\xi\cdot(x-y)}\,dx\,dy$$

Make the change of variables $z = x - y$:

$$\begin{aligned} &= \int_{\mathbb{R}^n} g(y)e^{-2\pi i\xi\cdot y} \int_{\mathbb{R}^n} f(z)e^{-2\pi i\xi\cdot z}\,dz\,dy \\ &= \int_{\mathbb{R}^n} g(y)e^{2\pi i\xi\cdot y}\widehat{f}(\xi)\,dy \\ &= \widehat{f}(\xi)\widehat{g}(\xi). \end{aligned}$$

2. We have $|\widehat{f}| \leq \|f\|_1$. If $h \in \mathbb{R}^n$,

$$\widehat{f}(\xi + h) = \int_{\mathbb{R}^n} f(x)e^{2\pi i\xi\cdot x}e^{2\pi ih\cdot x}\,dx,$$

so

$$|\widehat{f}(\xi + h) - \widehat{f}(\xi)| \leq \int_{\mathbb{R}^n} |f(x)||e^{-2\pi ih\cdot x} - 1|\,dx$$

$|f||e^{-2\pi i\xi\cdot x} - 1| \leq 2|f| \in L^2$, so we may apply the dominated convergence theorem to conclude that

$$\limsup_{h\to 0} |\widehat{f}(\xi + h) - \widehat{f}(\xi)| \leq \int_{\mathbb{R}^n} \limsup_{h\to 0} |e^{-2\pi ih\cdot x} - 1||f(x)|\,dx = 0.$$

3. Let $\xi \in \mathbb{R}^n$. Then

$$(\widehat{f})(\xi) = \widehat{f}(\xi - \eta) = \int_{\mathbb{R}^n} e^{-2\pi i(\xi-\eta)\cdot x} f(x)\,dx = \int_{\mathbb{R}^n} e^{2\pi i\xi\cdot x}E_k(x)f(x)\,dx = \widehat{E_k f}(\xi).$$

4.

$$\widehat{f \circ T}(\xi) = \int_{\mathbb{R}^n} f \circ T(x)e^{-2\pi i\xi\cdot x}\,dx$$

Make the change of variables $y = Tx$, so $x = Sy$ and $dx = |\det(S)|\,dy$.

$$= \int_{\mathbb{R}^n} f(y)e^{-2\pi i\xi\cdot Sy}|\det(S)|\,dy$$

Use the fact that $a \cdot (Sb) = S^\top a \cdot b$:

$$\begin{aligned} &= \int_{\mathbb{R}^n} f(y) e^{-2\pi i S^\top \xi \cdot y} |\det(S)|\, dy \\ &= |\det(S)| \widehat{f} \circ S^\top(\xi). \end{aligned}$$

5. Set $Tx = x/t$. so $Sy = ty$. Define $p_t(f)(x) = t^{-n} f(x/t) = |\det(S)|^{-1} f \circ T(x)$. By the previous part,

$$\widehat{O_t(f)} = \frac{1}{|\det(S)|} \widehat{f \circ T} = \frac{1}{|\det(S)|} |\det(S)| \widehat{f} \circ S^\top = \widehat{f}(t\xi) = t^n O_{1/t} \circ \widehat{f}. \qquad \square$$